

Maximal Blaschke products

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Abstract. We consider the classical problem of maximizing the derivative at a fixed point over the set of all bounded analytic functions in the unit disk with prescribed critical points. We show that the extremal function is essentially unique and always an indestructible Blaschke product. This result extends the Nehari–Schwarz Lemma and leads to a new class of Blaschke products called maximal Blaschke products. We establish a number of properties of maximal Blaschke products, which indicate that maximal Blaschke products constitute an appropriate infinite generalization of the class of finite Blaschke products.

1 Introduction and Results

Let H^∞ denote the space of all functions analytic and bounded in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ equipped with the norm

$$\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)| < \infty, \quad f \in H^\infty.$$

A sequence $\mathcal{C} = (z_j)$ in \mathbb{D} is called an H^∞ critical set, if there exists a nonconstant function f in H^∞ whose critical points are precisely the points on the sequence \mathcal{C} counting multiplicities. This means that if the point z_j occurs m times in the sequence, then f' has a zero at z_j of precise order m , and $f'(z) \neq 0$ for every $z \in \mathbb{D} \setminus \mathcal{C}$. For an H^∞ critical set \mathcal{C} we define the subspace

$$\mathcal{F}_{\mathcal{C}} := \{f \in H^\infty : f'(z) = 0 \text{ for any } z \in \mathcal{C}\}$$

of all functions $f \in H^\infty$ such that any point of the sequence \mathcal{C} is a critical point of f (with at least the prescribed multiplicity).

Our first theorem shows that the set $\mathcal{F}_{\mathcal{C}}$ always contains a Blaschke product whose critical set is precisely the sequence \mathcal{C} . In fact, more is true:

Theorem 1.1

Let $\mathcal{C} = (z_j)$ be an H^∞ critical set and let N denote the number of times that 0 appears in the sequence \mathcal{C} . Then the extremal problem

$$\max \left\{ \operatorname{Re} f^{(N+1)}(0) : f \in \mathcal{F}_{\mathcal{C}}, \|f\|_\infty \leq 1 \right\} \quad (1.1)$$

has a unique solution $B_{\mathcal{C}} \in \mathcal{F}_{\mathcal{C}}$. The extremal function $B_{\mathcal{C}}$ is an indestructible Blaschke product with critical set \mathcal{C} and is normalized by $B_{\mathcal{C}}(0) = 0$ and $B_{\mathcal{C}}^{(N+1)}(0) > 0$. If \mathcal{C} is a finite sequence consisting of m points, then $B_{\mathcal{C}}$ is a finite Blaschke product of degree $m + 1$; otherwise, $B_{\mathcal{C}}$ is an infinite Blaschke product.

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Note that Theorem 1.1 says that the critical points of the extremal function B_C are *exactly* the points of the sequence C with prescribed multiplicity, so there are no “extra critical points” and C is the critical set of B_C .

The crucial part of Theorem 1.1 is the assertion that the extremal function B_C is always an *indestructible Blaschke product*. Recall that a Blaschke product is called indestructible (see [28, 35]) if for any conformal automorphism T of the unit disk the composition $T \circ B_C$ is again a Blaschke product. Note that postcomposition by such a conformal automorphism does not change the critical set of a function in H^∞ . Therefore, for any conformal automorphism T of \mathbb{D} , we call the Blaschke product $T \circ B_C$ a *maximal Blaschke product with critical set C* .

If $N = 0$, then the extremal problem (1.1) is exactly the problem of maximizing the derivative at a point, i.e., exactly the character of Schwarz’ lemma. Let us put this observation in perspective.

Remark 1.2 (The Nehari–Schwarz Lemma)

In the special case where C is a *finite* sequence, Theorem 1.1 is essentially the classical and well-known Nehari–Schwarz lemma.

- (a) In fact, if $C = \emptyset$, then $\mathcal{F}_C = H^\infty$, so all bounded analytic functions are competing functions, and Theorem 1.1 is just the statement of Schwarz’ lemma, which implies that B_\emptyset is the identity map. In particular, the maximal Blaschke products without critical points, i.e., the locally univalent maximal Blaschke products are precisely the unit disk automorphisms.
- (b) If $C \neq \emptyset$ is a finite sequence and $N = 0$, then Theorem 1.1 is exactly Nehari’s 1947 generalization of Schwarz’ lemma (see [31], Corollary¹ to Theorem 1). In particular, if $C = (z_1, \dots, z_m)$ is a finite sequence consisting of m points, then every maximal Blaschke product with critical set C is a finite Blaschke product of degree $m + 1$. As we shall see in Remark 3.2 below, the converse is also true. Hence the maximal Blaschke products with finitely many critical points are precisely the finite Blaschke products.

By these remarks, Theorem 1.1 might be viewed as an extension of the Nehari–Schwarz Lemma to arbitrarily many critical points.

We now return to the extremal problem (1.1) and to a discussion of maximal Blaschke products and their properties. As we shall see, maximal Blaschke products do have similar characteristics as Bergman space inner functions on the one hand and display many properties of finite Blaschke products on the other hand.

We begin by relating maximal Blaschke products with Bergman space inner functions. For this we note that a similar type of extremal problem as (1.1) was considered before

¹In his formulation of this Corollary, Nehari apparently assumes, implicitly, that the origin is not a critical point. Otherwise, Nehari’s statement concerning the case of equality would not be entirely correct.

for various classes of analytic functions such as Hardy spaces and Bergman spaces, but *with prescribed zeros instead of prescribed critical points*. The following remark describes this connection in full detail.

Remark 1.3 (Hardy spaces and Bergman spaces)

If the sequence \mathcal{C} is the critical set of a bounded analytic function and if N denotes the multiplicity of the point 0 in \mathcal{C} , then according to Theorem 1.1 the maximal Blaschke product $B_{\mathcal{C}}$ with critical set \mathcal{C} normalized by $B_{\mathcal{C}}(0) = 0$ and $B_{\mathcal{C}}^{(N+1)}(0) > 0$ is the unique solution to the extremal problem

$$\max \left\{ \operatorname{Re} f^{(N+1)}(0) : f \in H^\infty, \|f\|_\infty \leq 1 \text{ and } f'(z) = 0 \text{ for } z \in \mathcal{C} \right\}.$$

This extremal property of a maximal Blaschke product is reminiscent of a well-known extremal property of

(i) Blaschke products in the Hardy spaces H^∞ and

$$H^p := \left\{ f : \mathbb{D} \rightarrow \mathbb{C} \text{ analytic} : \|f\|_p := \left(\lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p} < +\infty \right\},$$

where $1 < p < +\infty$,

as well as the

(ii) canonical divisors in the (weighted) Bergman spaces

$$\mathcal{A}_\alpha^p = \left\{ f : \mathbb{D} \rightarrow \mathbb{C} \text{ analytic} : \|f\|_{p,\alpha} := \left(\frac{1}{\pi} \iint_{\mathbb{D}} (1 - |z|^2)^\alpha |f(z)|^p d\sigma_z \right)^{1/p} < +\infty \right\},$$

where $-1 < \alpha < +\infty$, $1 < p < +\infty$ and $d\sigma_z$ denotes two-dimensional Lebesgue measure with respect to z ,

when the *zeros* are prescribed.

More precisely, let $1 < p \leq +\infty$, let the sequence $\mathcal{C} = (z_j)$ in \mathbb{D} be the *zero set* of a function in H^p and let N denote the multiplicity of the point 0 in \mathcal{C} . Then the (unique) solution to the extremal problem

$$\max \left\{ \operatorname{Re} f^{(N)}(0) : f \in H^p, \|f\|_p \leq 1 \text{ and } f(z) = 0 \text{ for } z \in \mathcal{C} \right\}$$

is a Blaschke product B with zero set \mathcal{C} normalized by $B^{(N)}(0) > 0$, see [13, §5.1].

Hedenmalm [17] (see also [10, 11]) had the idea of posing an appropriate counterpart of the latter extremal problem for Bergman spaces. His goal was to find a faithful analogue of Blaschke products in Bergman spaces. As before, let $\mathcal{C} = (z_j)$ be a sequence in \mathbb{D} where the point 0 occurs N times and assume that \mathcal{C} is the zero set of a function in \mathcal{A}_α^p . Then the extremal problem

$$\max \left\{ \operatorname{Re} f^{(N)}(0) : f \in \mathcal{A}_\alpha^p, \|f\|_{p,\alpha} \leq 1 \text{ and } f(z) = 0 \text{ for } z \in \mathcal{C} \right\}$$

has a unique extremal function $\mathcal{G} \in \mathcal{A}_\alpha^p$, which vanishes precisely on \mathcal{C} and is normalized by $\mathcal{G}^{(N)}(0) > 0$. The function \mathcal{G} is called the canonical divisor for \mathcal{C} or a Bergman space inner function. These functions play an extraordinary rôle in the modern theory of Bergman spaces, see [18, 13].

In summary, we have the following situation:

prescribed data	function space	extremal function
critical set \mathcal{C}	H^∞	maximal Blaschke product with critical set \mathcal{C}
zero set \mathcal{C}	H^p	Blaschke product with zero set \mathcal{C}
zero set \mathcal{C}	\mathcal{A}_α^p	canonical divisor with zero set \mathcal{C}

In light of this strong analogy, one expects that maximal Blaschke products enjoy similar properties as Blaschke products in H^p spaces and canonical divisors in Bergman spaces, with the critical points playing the rôle of the zeros. An example is analytic continuability. It is a familiar result in H^p theory that a Blaschke product has a holomorphic extension across every open arc of $\partial\mathbb{D}$ which does not contain any limit point of its zero set, see [14, Chapter II, Theorem 6.1]. The same is true for a canonical divisor in Bergman spaces. This was proved by Sundberg [40] in 1997, who improved earlier work of Duren, Khavinson, Shapiro and Sundberg [10, 11] and Duren, Khavinson and Shapiro [12]. Now keeping in mind that the critical points of maximal Blaschke products take the rôle of the zeros of Blaschke products and canonical divisors respectively, one expects that a maximal Blaschke product has an analytic continuation across every open arc of $\partial\mathbb{D}$ which does not meet any limit point of its critical set. This in fact turns out to be true:

Theorem 1.4 (Analytic continuability of maximal Blaschke products)

Let $F : \mathbb{D} \rightarrow \mathbb{D}$ be a maximal Blaschke product with critical set \mathcal{C} . Then F has an analytic continuation across any arc of $\partial\mathbb{D}$ which is free of limit points of \mathcal{C} .

Since a Blaschke product has an analytic continuation to a point $\xi \in \partial\mathbb{D}$ if and only if ξ is not a limit point of its zeros, Theorem 1.4 leads to the following conclusion.

Corollary 1.5

The limit points of the critical set of a maximal Blaschke product coincide with the limit points of its zero set.

We next turn to a result about the structural properties of H^∞ critical sets. It follows from the results in [23] that the union of two H^∞ critical sets is not necessarily an H^∞ critical set. However, if the two H^∞ critical sets have no common accumulation point on the unit circle, then their union is again an H^∞ critical set as the following result shows.

Theorem 1.6

Let \mathcal{C}_1 and \mathcal{C}_2 be two H^∞ critical sets such that $\overline{\mathcal{C}_1} \cap \overline{\mathcal{C}_2} \cap \partial\mathbb{D} = \emptyset$. Then $\mathcal{C}_1 \cup \mathcal{C}_2$ is an H^∞ critical set.

The analogous statement for the zero sets of Bergman space inner functions is due to Sundberg [40].

We next shift attention to maximal Blaschke products as generalizations of finite Blaschke products. Recall that in view of Remark 1.2 (b) every finite Blaschke product is a maximal Blaschke product. A rather strong property of the class of finite Blaschke products is their semigroup property with respect to composition. In contrast, the composition of two Blaschke products does not need to be a Blaschke product (just consider non-indestructible Blaschke products). However, in case of maximal Blaschke products the following result holds.

Theorem 1.7 (Semigroup property of maximal Blaschke products)

The set of maximal Blaschke products is closed under composition.

Finally, we consider the boundary behaviour of maximal Blaschke products. Heins [20] showed that a function $B \in H^\infty$ is a *finite* Blaschke product if and only if

$$\lim_{z \rightarrow \zeta} (1 - |z|^2) \frac{|B'(z)|}{1 - |B(z)|^2} = 1 \quad (1.2)$$

for every $\zeta \in \partial\mathbb{D}$. The next theorem gives a partial extension of this result for maximal Blaschke products.

Theorem 1.8 (Boundary behaviour of maximal Blaschke products)

Let B be a maximal Blaschke product with critical set \mathcal{C} . Then (1.2) holds for every $\zeta \in \partial\mathbb{D} \setminus \overline{\mathcal{C}}$.

The results of the present paper are obtained by using the equivalence of two types of sets, critical sets for H^∞ and zero sets for conformal Riemannian pseudometrics with curvature at most -4 , see Corollary 2.7 below. It turns out that the latter zero sets are simpler to work with. Accordingly, we start in Section 2 by describing the relation between bounded analytic functions and negatively curved conformal pseudometrics. Section 3 contains the proofs of the main results of this paper. It also gives a characterization of maximal Blaschke products in terms of “maximal” conformal pseudometrics. In Section 4, we generalize our results to analytic functions defined on simply connected proper subdomains of the complex plane other than the unit disk by using the Riemann mapping theorem. There, we also indicate a connection between maximal Blaschke products and the well-known Ahlfors’ map for domains of finite connectivity $n \geq 2$. We close the paper with a final Section 5, which presents a number of open problems.

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2 Auxiliary results

The proofs in this paper are based on conformal (Riemannian) pseudometrics and rely in particular on the results of [23]. We first give a quick account of the relevant facts from conformal geometry and refer to [4, 19, 21, 22, 23, 24, 39] for more information.

We call a continuous nonnegative function $\lambda : G \rightarrow \mathbb{R}$ defined on a domain $G \subseteq \mathbb{C}$ a conformal density and the quantity $\lambda(z) |dz|$ a conformal pseudometric on G . We say $\lambda(z) |dz|$ has a zero of order $m_0 > 0$ at $z_0 \in G$ if

$$\lim_{z \rightarrow z_0} \frac{\lambda(z)}{|z - z_0|^{m_0}} \quad \text{exists and} \neq 0.$$

In this paper we will only consider conformal pseudometrics $\lambda(z) |dz|$ with isolated zeros. A sequence $\mathcal{C} = (\xi_j) \subset G$

$$(\xi_j) := (\underbrace{z_1, \dots, z_1}_{m_1\text{-times}}, \underbrace{z_2, \dots, z_2}_{m_2\text{-times}}, \dots), \quad z_k \neq z_n \text{ if } k \neq n,$$

is called the zero set of a conformal pseudometric $\lambda(z) |dz|$, if $\lambda(z) > 0$ for $z \in G \setminus \mathcal{C}$ and if $\lambda(z) |dz|$ has a zero of order $m_k \in \mathbb{N}$ at z_k for all k . We will always assume that λ is of class C^2 in a neighborhood of any point $z_0 \in G$ where $\lambda(z_0) > 0$. Hence the curvature κ_λ of $\lambda(z) |dz|$ can be defined by

$$\kappa_\lambda(z_0) = -\frac{\Delta(\log \lambda)}{\lambda^2}(z_0) \tag{2.1}$$

for any point $z_0 \in G$ with $\lambda(z_0) > 0$.

An important aspect of curvature is its conformal invariance. Suppose $f : G \rightarrow D$ is an analytic function from a domain G into a domain D and let D be equipped with a positive conformal pseudometric $\lambda(z) |dz|$ with curvature κ_λ . Then the pullback to G via f of $\lambda(z) |dz|$ is a conformal pseudometric on D defined by

$$f^* \lambda(z) |dz| := \lambda(f(z)) |dz|.$$

Now the curvature of the pullback pseudometric $f^* \lambda(z) |dz|$ and the pseudometric $\lambda(w) |dw|$ are related by the fundamental identity

$$\kappa_{f^* \lambda}(z) = \kappa_\lambda(f(z)),$$

which is valid for every $z \in G \setminus \{z \in G : f'(z) = 0\}$. Note that in particular the zero set of the pseudometric $f^* \lambda(z) |dz|$ is precisely the critical set of the function f .

The ubiquitous example for a conformal pseudometric is the Poincaré metric or hyperbolic metric

$$\lambda_{\mathbb{D}}(z) |dz| = \frac{1}{1 - |z|^2} |dz|$$

for the unit disk \mathbb{D} ; it has constant curvature -4 on \mathbb{D} . The hyperbolic metric $\lambda_{\mathbb{D}}(z) |dz|$ has the following important property.

Theorem 2.1

Let $\lambda(z)|dz|$ be a conformal pseudometric on \mathbb{D} with curvature $\kappa_\lambda(z) \leq -4$ for all $z \in \mathbb{D}$ with $\lambda(z) > 0$. Then the following statements hold.

(a) For every $z \in \mathbb{D}$

$$\lambda(z) \leq \lambda_{\mathbb{D}}(z).$$

(b) If $\lambda(z_0) = \lambda_{\mathbb{D}}(z_0)$ for some $z_0 \in \mathbb{D}$, then $\lambda(z) = \lambda_{\mathbb{D}}(z)$ for all $z \in \mathbb{D}$.

Theorem 2.1 (a) is due to Ahlfors [1] and it is usually called Ahlfors' lemma. The case of equality in Ahlfors' lemma, i.e. Theorem 2.1 (b), was proved by Heins [19, §7], see also Royden [36] and Minda [29].

The following theorem gives a sharpening of Theorem 2.1 for conformal pseudometrics with prescribed zero set.

Theorem 2.2 (Maximal conformal pseudometric)

Let C be a sequence of points in \mathbb{D} such that there exists a conformal pseudometric in \mathbb{D} with zero set C and curvature ≤ -4 in $\mathbb{D} \setminus C$. Then there exists a unique conformal pseudometric $\lambda_{\max}(z)|dz|$ on \mathbb{D} with zero set C and constant curvature -4 on $\mathbb{D} \setminus C$ such that for any conformal pseudometric $\lambda^*(z)|dz|$ with zero set $C^* \supseteq C$ and curvature $\kappa_{\lambda^*}(z) \leq -4$ on $\mathbb{D} \setminus C^*$ the following conditions hold.

(a) For every $z \in \mathbb{D}$

$$\lambda^*(z) \leq \lambda_{\max}(z).$$

(b) If

$$\lim_{z \rightarrow z_0} \frac{\lambda^*(z)}{\lambda_{\max}(z)} = 1$$

for some point $z_0 \in \mathbb{D}$, then $\lambda^*(z) = \lambda_{\max}(z)$ for all $z \in \mathbb{D}$.

We note that the first statement of Theorem 2.2 is a result by Heins [19, Theorem 13.1]. It suggests calling the conformal pseudometric $\lambda_{\max}(z)|dz|$ the maximal conformal pseudometric on \mathbb{D} with zero set C and curvature -4 on $\mathbb{D} \setminus C$.

Theorem 2.2 (b) will play a crucial rôle in the proof of Theorem 1.1. In order to prove it, we need the following result.

Lemma 2.3

Let G be a bounded and regular domain², let $b : \partial G \rightarrow (0, +\infty)$ be a continuous function on the boundary ∂G of G and let κ be a bounded, nonpositive and locally Hölder continuous on G . Then there exists a unique positive conformal pseudometric $\lambda(z)|dz|$ on G with curvature $\kappa_\lambda \equiv \kappa$ on G such that λ is continuous on the closure \overline{G} and $\lambda(\xi) = b(\xi)$ for all $\xi \in \partial G$.

²i.e. there exists a Green's function for G which vanishes continuously on the boundary of G .

For completeness, we sketch a proof of Lemma 2.3 here using a standard result from nonlinear elliptic PDEs.

Proof. We first note that the Dirichlet problem

$$\begin{aligned} \Delta u &= -\kappa(z) e^{2u} & \text{if } z \in G, \\ u(\xi) &= \log b(\xi) & \text{if } \xi \in \partial G. \end{aligned} \quad (2.2)$$

has a unique solution $u \in C^2(G) \cap C(\overline{G})$, see [15, p. 53–55 & p. 304] and [9, p. 286]. Taking into account the definition of curvature, i.e. the formula (2.1), this means that

$$\lambda(z) |dz| := e^{u(z)} |dz|$$

is the unique positive conformal pseudometric on G with curvature κ such that λ is continuous on G and $\lambda(\xi) = b(\xi)$ for every $\xi \in \partial G$. \blacksquare

Remark 2.4

The uniqueness statement in Lemma 2.3 ultimately comes from the maximum principle for solutions to the Dirichlet problem (2.2), see [15, Theorem 10.1]. In the terminology of conformal pseudometrics, the maximum principle says the following. Let $\lambda(z) |dz|$ and $\mu(z) |dz|$ be positive conformal pseudometrics on a bounded domain G with the following properties: $\kappa_\lambda \leq \kappa_\mu \leq 0$ on G , λ and μ are continuous and positive on \overline{G} and $\lambda(\xi) \leq \mu(\xi)$ for all $\xi \in \partial G$. Then $\lambda \leq \mu$ throughout G .

Before passing to the proof of Theorem 2.2 (b), we note that the special case of Theorem 2.2 (b) as described in Theorem 2.1 (b) allows for a quick proof using the so-called strong maximum principle of E. Hopf from the theory of elliptic PDEs, see e.g. [15, 29]. In the general case, we have to deal with the problem that the assumption

$$\lim_{z \rightarrow z_0} \frac{\lambda^*(z)}{\lambda_{max}(z)} = 1$$

in Theorem 2.2 (b) might hold for a point $z_0 \in \mathbb{D}$ where λ_{max} vanishes. As we shall see, Lemma 2.3 will enable us to still make use of Hopf's strong maximum principle.

Proof of Theorem 2.2 (b). We divide the proof in two cases.

(i) First assume that $\lambda^*(z_1) = \lambda_{max}(z_1)$ for some point $z_1 \in G := \mathbb{D} \setminus \mathcal{C}^*$. Then the function

$$u(z) := \log \frac{\lambda_{max}(z)}{\lambda^*(z)}$$

is twice continuously differentiable and nonnegative on G and

$$\begin{aligned} \Delta u(z) &= \Delta \log \lambda_{max}(z) - \Delta \log \lambda^*(z) \leq 4\lambda_{max}(z)^2 - 4\lambda^*(z)^2 \\ &= 4\lambda_{max}(z)^2(1 - e^{-2u(z)}) \leq 8\lambda_{max}(z)^2 u(z) \end{aligned}$$

for every $z \in G$. The strong maximum principle ([15, Theorem 3.5]) implies that $u \equiv 0$ in G , i.e. $\lambda^*(z) = \lambda_{max}(z)$ for all $z \in \mathbb{D}$ by continuity.

(ii) Now consider the case that $\lambda^*(z) < \lambda_{max}(z)$ for all $z \in \mathbb{D} \setminus \mathcal{C}^*$. This implies $\lambda^*(z) < \lambda_{max}(z)$ for all $z \in \mathbb{D} \setminus \mathcal{C}$. In particular, the point z_0 belongs to \mathcal{C} . Let m denote the order of the zero of $\lambda_{max}(z) |dz|$ at z_0 . Since \mathcal{C} is a discrete subset of the unit disk \mathbb{D} , there exists an open disk K compactly contained in \mathbb{D} such that $z_0 \in K$, $\lambda^*(z) > 0$ for every $z \in \overline{K} \setminus \{z_0\}$ and

$$\lambda^*(\xi) < \lambda_{max}(\xi) \quad \text{for all } \xi \in \partial K. \quad (2.3)$$

We consider the pseudometric

$$\tilde{\lambda}_{max}(z) |dz| := \frac{\lambda_{max}(z)}{|z - z_0|^m} |dz|$$

on the punctured disk $K \setminus \{z_0\}$. Clearly, $\tilde{\lambda}_{max}$ is of class C^2 in $K \setminus \{z_0\}$ and a quick computation shows that $\tilde{\lambda}_{max}(z) |dz|$ has curvature $\kappa(z) = -4|z - z_0|^{2m}$ there. Since z_0 is a zero of $\lambda_{max}(z) |dz|$ of order m , the density $\tilde{\lambda}_{max}$ has a continuous extension to K . We claim that (the continuous extension of) $\tilde{\lambda}_{max}$ is of class C^2 on the entire disk K . In fact, Lemma 2.3 shows that there is a positive conformal pseudometric $\mu(z) |dz|$ on K with curvature κ there, continuous on \overline{K} and $\mu(\xi) = \tilde{\lambda}_{max}(\xi)$ for every $\xi \in \partial K$. In particular, μ is of class C^2 on K . It now turns out that $\mu \equiv \tilde{\lambda}$. To see this, consider the auxiliary function

$$v(z) := \max \left\{ 0, \log \frac{\tilde{\lambda}_{max}(z)}{\mu(z)} \right\}, \quad z \in K.$$

If $z_1 \in K$ such that $v(z_1) > 0$, then

$$\Delta v(z_1) = \Delta \log \tilde{\lambda}_{max}(z_1) - \Delta \log \mu(z_1) = 4|z_1 - z_0|^{2m} \tilde{\lambda}_{max}(z_1)^2 - 4|z_1 - z_0|^{2m} \mu(z_1)^2 \geq 0,$$

so v is a continuous subharmonic function on $\{z \in K : v(z) > 0\}$ with boundary values 0, i.e., $v \leq 0$ by the maximum principle for subharmonic functions. This shows that $v \equiv 0$, i.e., $\tilde{\lambda}_{max} \leq \mu$. A similar argument establishes the reverse inequality, so we have proved that $\tilde{\lambda}_{max} = \mu$ is of class C^2 on the entire disk K .

We now go back to inequality (2.3). Lemma 2.3 shows that there exists a positive pseudometric $\nu(z) |dz|$ on K with curvature $\kappa(z) = -4|z - z_0|^{2m}$ on K such that ν is continuous on \overline{K} and

$$\frac{\lambda^*(\xi)}{|\xi - z_0|^m} < \nu(\xi) < \frac{\lambda_{max}(\xi)}{|\xi - z_0|^m} = \tilde{\lambda}_{max}(\xi) \quad \text{for every } \xi \in \partial K.$$

Then

$$\frac{\lambda^*(z)}{|z - z_0|^m} \leq \nu(z) \leq \tilde{\lambda}_{max}(z) \quad \text{for all } z \in K \setminus \{z_0\}. \quad (2.4)$$

The right inequality follows directly from the maximum principle (see Remark 2.4) applied to the conformal pseudometrics $\nu(z) |dz|$ and $\tilde{\lambda}_{max}(z) |dz|$. In order to prove the left inequality in (2.4), consider the function $s : K \setminus \{z_0\} \rightarrow \mathbb{R}$ defined by

$$s(z) := \max \left\{ 0, \log \left(\frac{\lambda^*(z)}{|z - z_0|^m \nu(z)} \right) \right\}.$$

If $z \in K \setminus \{z_0\}$ such that $s(z) > 0$, then

$$\Delta s(z) = \Delta \log \lambda^*(z) - \Delta \log \nu(z) \geq 4\lambda^*(z)^2 - 4|z - z_0|^{2m}\nu(z)^2 \geq 0.$$

Hence, the nonnegative function $s : K \setminus \{z_0\} \rightarrow \mathbb{R}$ is subharmonic on $K \setminus \{z_0\}$. Moreover, since $\lambda^*(z)$ has a zero at z_0 of at least order m , the function s has a continuous extension to the point z_0 . As a consequence, the function s has a subharmonic extension to the entire disk K with boundary values 0. Thus the maximum principle for subharmonic functions leads to $s \leq 0$ throughout K , so, by the definition of the function s , the left inequality in (2.4) follows.

We next combine the assumption of Theorem 2.2 (b) with the estimates (2.4) and obtain

$$\lim_{z \rightarrow z_0} \frac{\tilde{\lambda}_{max}(z)}{\nu(z)} = \lim_{z \rightarrow z_0} \frac{\lambda_{max}(z)}{\lambda^*(z)} = 1.$$

Since, as we have observed above, the functions ν and $\tilde{\lambda}_{max}$ are of class C^2 on K , this means that the auxiliary function

$$\tilde{u}(z) := \log \frac{\tilde{\lambda}_{max}(z)}{\nu(z)}$$

is nonnegative and of class C^2 in K with $\tilde{u}(z_0) = 0$. Now, similar to part (a), one can show that

$$\Delta \tilde{u}(z) \leq 8\lambda_{max}(z)^2 |z - z_0|^{2m} \tilde{u}(z), \quad z \in K.$$

Hence the strong maximum principle gives $\tilde{u} \equiv 0$, contradicting the boundary condition $\tilde{u}(\xi) > 0$ for every $\xi \in \partial K$. We conclude that our assumption $\lambda^* \not\equiv \lambda_{max}$ cannot hold and the proof of Theorem 2.2 (b) is complete. \blacksquare

The next result shows that we can represent the maximal pseudometric $\lambda_{max}(z) |dz|$ in Theorem 2.2 as the pullback of the Poincaré metric $\lambda_{\mathbb{D}}(z) |dz|$ under a specific function $F \in H^\infty$.

Theorem 2.5

Let C be an H^∞ critical set and let $\lambda_{max}(z) |dz|$ be the maximal conformal pseudometric on \mathbb{D} with zero set C and curvature -4 on $\mathbb{D} \setminus C$. Then the following statements hold.

(a) *There exists a function $F \in H^\infty$ with critical set C such that*

$$\lambda_{max}(z) = \frac{|F'(z)|}{1 - |F(z)|^2}, \quad z \in \mathbb{D}. \quad (2.5)$$

Moreover, F is uniquely determined by $\lambda_{max}(z) |dz|$ up to postcomposition with a unit disk automorphism.

(b) *The function F in (a) is an indestructible Blaschke product with critical set C .*

Remark 2.6

- (a) Statement (a) and part of statement (b) of Theorem 2.5 can be found in the work of Heins [19, Theorem 29.1]. His proof is in three steps. In a first step, Heins proved that for a finite sequence \mathcal{C} (consisting of m points) in \mathbb{D} there is always a finite Blaschke product of degree $m + 1$ with critical set \mathcal{C} and that this finite Blaschke product is uniquely determined by its critical set \mathcal{C} up to postcomposition by a unit disk automorphism. His second step consists of showing that if B is a finite Blaschke product (of degree $m + 1$) with critical set \mathcal{C} (then containing exactly m points, cf. [38, p. 78]) then the pullback of $\lambda_{\mathbb{D}}(z) |dz|$ via B ,

$$\frac{|B'(z)|}{1 - |B(z)|^2} |dz|$$

is the maximal conformal pseudometric $\lambda_{\max}(z) |dz|$ on \mathbb{D} with zero set \mathcal{C} and curvature -4 on $\mathbb{D} \setminus \mathcal{C}$. Finally, part (a) of Theorem 2.5 is established by letting m tend to ∞ .

We note that the first and the second step together prove Theorem 2.5 (b) for *finite* sequences \mathcal{C} . The general case of Theorem 2.5 (b) is more involved and is proved in [23].

- (b) Theorem 2.5 (a) is actually a special instance of “Liouville’s Theorem”:

Every conformal pseudometric $\lambda(z) |dz|$ in \mathbb{D} with zero set \mathcal{C} and constant curvature -4 on $\mathbb{D} \setminus \mathcal{C}$ can be written as the pullback of the hyperbolic metric $\lambda_{\mathbb{D}}(w) |dw|$ under a function $f \in H^{\infty}$ with critical set \mathcal{C} , that is

$$\lambda(z) = \frac{|f'(z)|}{1 - |f(z)|^2}, \quad z \in \mathbb{D}. \quad (2.6)$$

Moreover, the analytic function $f : \mathbb{D} \rightarrow \mathbb{D}$ (the so-called developing map of $\lambda(z) |dz|$) is uniquely determined by the pseudometric $\lambda(z) |dz|$ up to postcomposition with a unit disk automorphism.

In fact, Liouville [27] stated only the zerofree case ($\mathcal{C} = \emptyset$) and his proof is not entirely convincing by today’s standards. A very elegant geometric proof of the zerofree case was given by D. Minda in his notes [30]. The general case of Liouville’s Theorem ($\mathcal{C} \neq \emptyset$) can be found e.g. in [6, 7, 8, 24, 33, 41]. Note that in Liouville’s theorem, the zeros of the pseudometric $\lambda(z) |dz|$ are precisely the critical points of its developing map $f \in H^{\infty}$.

- (c) Using the terminology of Liouville’s theorem, Theorem 2.5 (b) says that for a sequence \mathcal{C} in \mathbb{D} the developing maps of the maximal conformal pseudometric in \mathbb{D} with zero set \mathcal{C} and curvature -4 in $\mathbb{D} \setminus \mathcal{C}$ are indestructible Blaschke products with critical set \mathcal{C} .

The next result provides a simple criterion for a sequence \mathcal{C} being an H^∞ critical set, which will be useful later on.

Corollary 2.7

Let \mathcal{C} be a sequence of points in \mathbb{D} . Then the following conditions are equivalent.

- (a) \mathcal{C} is an H^∞ critical set.
- (b) \mathcal{C} is the zero set of a conformal pseudometric in \mathbb{D} with zero set \mathcal{C} and curvature ≤ -4 in $\mathbb{D} \setminus \mathcal{C}$.

Proof. Let \mathcal{C} be the critical set of $f \in H^\infty$. Then the pullback

$$f^* \lambda_{\mathbb{D}}(z) |dz| = \frac{|f'(z)|}{1 - |f(z)|^2} |dz|$$

of the hyperbolic metric $\lambda_{\mathbb{D}}(w) |dw|$ under f is a conformal pseudometric with zero set \mathcal{C} and curvature -4 on $\mathbb{D} \setminus \mathcal{C}$. This proves “(a) \implies (b)”. Conversely, if (b) holds, then Theorem 2.2 implies that there is a maximal conformal pseudometric $\lambda_{max}(z) |dz|$ with zero set \mathcal{C} and curvature -4 on $\mathbb{D} \setminus \mathcal{C}$. Using Theorem 2.5 (a), we see that there is a function $F \in H^\infty$ with critical set \mathcal{C} . ■

Remark 2.8

Heins [19, §25 & §26] initiated the study of the mapping properties of the developing maps of maximal conformal pseudometrics. He obtained some necessary conditions as well as sufficient conditions for developing maps of maximal conformal pseudometrics, but he did not prove that they are always Blaschke products. He also posed the problem of characterizing the developing maps of maximal conformal pseudometrics, cf. [19, §26 & §29].

3 Proofs

3.1 Proof of Theorem 1.1 and some consequences

The proof of Theorem 1.1 consists in identifying the extremal function(s) for the extremal problem (1.1) as the developing maps of the *maximal* conformal pseudometric $\lambda_{max}(z) |dz|$ with zero set \mathcal{C} and curvature -4 on $\mathbb{D} \setminus \mathcal{C}$. This is accomplished with the help of Theorem 2.2 (b) and an application of Theorem 2.5 (b).

Proof of Theorem 1.1.

(a) We first note that a normal family argument ensures the existence of an extremal function for the extremal problem (1.1), i.e., there is an analytic function $g \in \mathcal{F}_{\mathcal{C}}$ such that $\operatorname{Re} g^{(N+1)}(0) \geq \operatorname{Re} f^{(N+1)}(0)$ for all $f \in \mathcal{F}_{\mathcal{C}}$. It is easy to show that $g^{(N+1)}(0)$ is real and positive and that $g(0) = 0$.

In fact, since \mathcal{C} is an H^∞ critical set and N is the number of times that 0 occurs in the sequence \mathcal{C} , there exists some $f \in \mathcal{F}_{\mathcal{C}}$ with $f^{(N+1)}(0) \neq 0$. In view of $\eta f \in \mathcal{F}_{\mathcal{C}}$ for

any $f \in \mathcal{F}_C$ and any $\eta \in \partial\mathbb{D}$, this implies $g^{(N+1)}(0) = \operatorname{Re} g^{(N+1)}(0) > 0$. Since

$$\tilde{g}(z) := \frac{g(z) - g(0)}{1 - \overline{g(0)}g(z)} = \frac{g^{(N+1)}(0)}{1 - |g(0)|^2} \frac{1}{(N+1)!} z^{N+1} + \dots$$

belongs to \mathcal{F}_C , we deduce that $g(0) = 0$.

(b) Let

$$\lambda^*(z) |dz| := \frac{|g'(z)|}{1 - |g(z)|^2} |dz|, \quad z \in \mathbb{D},$$

be the pullback of the hyperbolic metric $\lambda_{\mathbb{D}}(z) |dz|$ under the holomorphic function g . Then $\lambda^*(z) |dz|$ is a conformal pseudometric on \mathbb{D} with zero set $\mathcal{C}^* \supseteq \mathcal{C}$ and curvature -4 on $\mathbb{D} \setminus \mathcal{C}^*$. By Theorem 2.2 there exists a maximal conformal pseudometric $\lambda_{\max}(z) |dz|$ on \mathbb{D} with zero set \mathcal{C} and curvature -4 in $\mathbb{D} \setminus \mathcal{C}$. Let $F \in H^\infty$ denote the developing map of $\lambda_{\max}(z) |dz|$ normalized by $F(0) = 0$ and $F^{(N+1)}(0) \geq 0$. Since \mathcal{C} is the critical set of F , it follows that $F^{(N+1)}(0) > 0$ and $F \in \mathcal{F}_C$, so

$$0 < F^{(N+1)}(0) \leq g^{(N+1)}(0). \quad (3.1)$$

On the other hand, the maximal property of $\lambda_{\max}(z) |dz|$ shows

$$\lambda^*(z) = \frac{|g'(z)|}{1 - |g(z)|^2} \leq \frac{|F'(z)|}{1 - |F(z)|^2} = \lambda_{\max}(z), \quad z \in \mathbb{D},$$

so

$$\frac{g^{(N+1)}(0)}{F^{(N+1)}(0)} = \lim_{z \rightarrow 0} \frac{\lambda^*(z)}{\lambda_{\max}(z)} \leq 1. \quad (3.2)$$

Conditions (3.1) and (3.2) together imply $F^{(N+1)}(0) = g^{(N+1)}(0)$ and, applying (3.2) again, we get that

$$\lim_{z \rightarrow 0} \frac{\lambda^*(z)}{\lambda_{\max}(z)} = 1.$$

Hence Theorem 2.2 (b) gives $\lambda^* \equiv \lambda_{\max}$ and Theorem 2.5 (a) shows that $g = T \circ F$ for some conformal automorphism T of the unit disk. Since $g(0) = F(0)$ and $g^{(N+1)}(0) = F^{(N+1)}(0)$, we finally arrive at $g = F$. In particular, g is uniquely determined.

(c) Theorem 2.5 (b) shows that $g = F$ is an indestructible Blaschke product. If \mathcal{C} is a finite sequence consisting of m points, then by a result of Heins [19] (see Remark 2.6 (a)), the function F in part (b) above is a finite Blaschke product of degree $m + 1$. If \mathcal{C} is an infinite sequence, then F cannot be a finite Blaschke product, since finite Blaschke products only have finitely many critical points, see e.g. [38, p. 78]. ■

We note the following immediate consequence of the above proof of Theorem 1.1.

Corollary 3.1

Let \mathcal{C} be a sequence of points in \mathbb{D} and $F \in H^\infty$ with critical set \mathcal{C} . Then the following are equivalent.

(a) *F is a maximal Blaschke product with critical set \mathcal{C} .*

(b) *The conformal pseudometric*

$$(F^* \lambda_{\mathbb{D}})(z) |dz| := \frac{|F'(z)|}{1 - |F(z)|^2} |dz|$$

is the maximal conformal pseudometric $\lambda_{max}(z) |dz|$ on \mathbb{D} with zero set C and curvature -4 on $\mathbb{D} \setminus C$.

Remark 3.2

As mentioned earlier (see Remark 1.2 (a)), any finite Blaschke product is a maximal Blaschke product. To see this, let F be a finite Blaschke product of degree $m + 1$ say, then F has a finite critical set $C = (z_1, \dots, z_m)$. By Remark 2.6 (a), $(F^* \lambda_{\mathbb{D}})(z) |dz|$ is the maximal conformal pseudometric on \mathbb{D} with zero set C and curvature -4 on $\mathbb{D} \setminus C$. So, Corollary 3.1 shows that F is a maximal Blaschke product with critical set C .

The next result follows easily from Theorem 1.1. It will be very useful later.

Corollary 3.3

Let (C_n) be a sequence of H^∞ critical sets with $\dots C_{n+1} \supseteq C_n \supseteq \dots \supseteq C_1$ and let F_n denote the extremal function for the extremal problem (1.1) for C_n . Assume that

$$C := \bigcup_{n=1}^{\infty} C_n$$

is an H^∞ critical set and let F be the extremal function for the extremal problem (1.1) for C . Then the sequence (F_n) converges to F locally uniformly in \mathbb{D} .

Proof. Let N_n denote the number of times that 0 appears in the sequence C_n . From $C_{n+1} \supseteq C_n$, we get $N_{n+1} \geq N_n$. Since $C = \bigcup_{n=1}^{\infty} C_n$ is an H^∞ critical set, we can assume that 0 occurs finitely often, say N times in the sequence C . This implies that $N_n = N$ for all but finitely many n , say for all $n \geq K$. Since $F \in \mathcal{F}_{C_n}$ for every positive integer n , we deduce $\operatorname{Re} F^{(N+1)}(0) \leq \operatorname{Re} F_n^{(N+1)}(0) \leq \operatorname{Re} F_{n+1}^{(N+1)}(0)$ for all $n \geq K$. If g is a subsequential limit function of the sequence (F_n) with respect to locally uniform convergence in \mathbb{D} , then this implies $\operatorname{Re} g^{(N+1)}(0) \geq \operatorname{Re} F^{(N+1)}(0)$. On the other hand, $g \in \mathcal{F}_C$, so $\operatorname{Re} g^{(N+1)}(0) \leq \operatorname{Re} F^{(N+1)}(0)$. By the uniqueness statement in Theorem 1.1, we deduce $g = F$. Since (F_n) is a normal family, we conclude that (F_n) has a unique subsequential limit function. This proves the corollary. ■

3.2 Proof of Theorem 1.4

We now shift attention to the proof of Theorem 1.4.

Proof of Theorem 1.4. Let m_j denote the number of times that z_j appears in the sequence $C = (z_j)$. For simplicity we set $z_1 = 0$ and $N = m_1$. We may assume that the maximal Blaschke product F for C is normalized by $F(0) = 0$ and $F^{(N+1)}(0) > 0$, so F is the unique extremal function for the extremal problem of Theorem 1.1.

Suppose that $z_0 \in \partial\mathbb{D}$ is not a limit point of \mathcal{C} . Then there exists an open disk K with the following three properties: $z_0 \in K$, the boundary ∂K intersects the unit circle perpendicularly and $\mathbb{D} \cap K$ contains none of the points of \mathcal{C} . We set $I := K \cap \partial\mathbb{D}$. It suffices to prove that F has an analytic continuation to $\mathbb{D} \cup K$.

Let F_n denote the extremal function for the extremal problem (1.1) for the finite critical set

$$\mathcal{C}_n := \left(\underbrace{0, \dots, 0}_{N\text{-times}}, \underbrace{z_2, \dots, z_2}_{m_2\text{-times}}, \dots, \underbrace{z_n, \dots, z_n}_{m_n\text{-times}} \right).$$

Note that for any positive integer n the function F_n is a finite Blaschke product of degree $N + m_2 + \dots + m_n + 1$ with $F_n(0) = 0$ and $F_n^{(N+1)}(0) > 0$. Since by construction $\cup_{n=1}^{\infty} \mathcal{C}_n = \mathcal{C}$, we see from Corollary 3.3 that $F_n \rightarrow F$ locally uniformly in \mathbb{D} . We now consider the auxiliary functions

$$\varphi_n(z) := \frac{F_n(z)}{zF_n'(z)}$$

for $n = 1, 2, \dots$. Since each F_n is a finite Blaschke product, the function φ_n is analytic in K and $0 < \varphi_n(\zeta) \leq 1$ for each $\zeta \in I$ by the Julia–Wolff–Carathéodory lemma, see [37]. We claim that (φ_n) is a normal family in K . Taking this for granted momentarily, we may assume that (φ_n) converges locally uniformly in K to a holomorphic limit function $\varphi : K \rightarrow \mathbb{C}$. It follows that

$$\varphi(z) = \frac{F(z)}{zF'(z)} \quad \text{for } z \in \mathbb{D} \cap K,$$

so $F(z)/(zF'(z))$ has an analytic continuation to K . In particular, the zeros of the Blaschke product F cannot accumulate on I , i.e. F has an analytic continuation across I , see [14, Chapter II, Theorem 6.1].

It remains to prove that the sequence (φ_n) is indeed a normal family in K . Since $\varphi_n(\zeta) \in \mathbb{R}$ for any $\zeta \in I$, we have

$$\varphi_n(\zeta) = \overline{\varphi_n(1/\bar{\zeta})} \quad \text{for all } \zeta \in K$$

by the Schwarz reflection principle. We now appeal to the Schwarz lemma

$$|F_n(z)| \leq |z|, \quad z \in \mathbb{D},$$

and the Schwarz–Pick lemma

$$|F_n'(z)| \leq \frac{1 - |F_n(z)|^2}{1 - |z|^2} \leq \frac{1}{1 - |z|^2}, \quad z \in \mathbb{D}.$$

Using $\log^+ x := \max\{\log x, 0\}$ for $x \in \mathbb{R}$, $x > 0$, these estimates lead to

$$\begin{aligned} \log^+ |\varphi_n(z)| &= \log^+ \left| \frac{F_n(z)}{zF_n'(z)} \right| \leq \log^+ \frac{1}{|F_n'(z)|} = \log^+ |F_n'(z)| - \log |F_n'(z)| \\ &\leq \log \left(\frac{1}{1 - |z|^2} \right) - \log |F_n'(z)|, \quad z \in \mathbb{D} \cap K, \end{aligned}$$

for $n = 1, 2, \dots$. By reflection, we get the similar estimate

$$\log^+ |\varphi_n(z)| \leq \log^+ \frac{1}{|F'_n(1/\bar{z})|}, \quad z \in K \setminus \mathbb{D},$$

for $n = 1, 2, \dots$. We now choose a compact subset Ω of K . Then $\Omega \subset K_R(0)$ for some $R > 1$ and

$$\begin{aligned} \iint_{\Omega} \log^+ |\varphi_n(z)| d\sigma_z &\leq \iint_{\Omega \cap \mathbb{D}} \log^+ \frac{1}{|F'_n(z)|} d\sigma_z + \iint_{\Omega \setminus \mathbb{D}} \log^+ \frac{1}{|F'_n(1/\bar{z})|} d\sigma_z \\ &\leq \iint_{\mathbb{D}} \log^+ \frac{1}{|F'_n(z)|} d\sigma_z + \iint_{1 < |z| < R} \log^+ \frac{1}{|F'_n(1/\bar{z})|} d\sigma_z \\ &\leq (1 + R^4) \iint_{\mathbb{D}} \log^+ \frac{1}{|F'_n(z)|} d\sigma_z \\ &\leq (1 + R^4) \iint_{\mathbb{D}} \log \left(\frac{1}{1 - |z|^2} \right) d\sigma_z - (1 + R^4) \iint_{\mathbb{D}} \log |F'_n(z)| d\sigma_z \\ &\leq c_1 - (1 + R^4) \iint_{\mathbb{D}} \log |F'_n(z)| d\sigma_z \end{aligned}$$

for some constant c_1 depending only on Ω . We note that $F'_n(z) = z^N g_n(z)$ with a holomorphic function $g_n : \mathbb{D} \rightarrow \mathbb{C}$ that satisfies $g_n(0) \neq 0$. Consequently,

$$\iint_{\Omega} \log^+ |\varphi_n(z)| d\sigma_z \leq c_2 - (1 + R^4) \iint_{\mathbb{D}} \log |g_n(z)| d\sigma_z, \quad n \geq 1,$$

where c_2 depends only on Ω . By the submean value inequality for subharmonic functions (see [34, Theorem 2.6.8]), we get

$$\iint_{\Omega} \log^+ |\varphi_n(z)| d\sigma_z \leq c_2 - (1 + R^4) \pi \log |g_n(0)|, \quad n \geq 1.$$

Since $g_n(0) = F_n^{(N+1)}(0)/(N+1)!$ and $F_n^{(N+1)}(0) \rightarrow F^{(N+1)}(0) > 0$ as $n \rightarrow +\infty$ we obtain

$$\iint_{\Omega} \log^+ |\varphi_n(z)| d\sigma_z \leq c_3 - (1 + R^4) \pi \log |F_n^{(N+1)}(0)| \leq c_4 < +\infty$$

for all $n \geq 1$, where c_3 and c_4 are some constants which do not depend on the functions φ_n . Thus, if $\Omega' \subset K$ is compact and

$$\text{dist}(z, \Omega') := \inf\{|z - z'| : z' \in \Omega'\}$$

denotes the euclidean distance of a point $z \in \mathbb{C}$ to Ω' , then there is a $\delta > 0$ such that the tubular neighborhood $\Omega_\delta := \{z \in \mathbb{C} : \text{dist}(z, \Omega') \leq \delta\}$ is entirely contained in K . Hence, there exists a constant \tilde{c} such that for any $z \in \Omega'$ and $n = 1, 2, \dots$,

$$\log^+ |\varphi_n(z)| \leq \frac{1}{\pi \delta^2} \iint_{|w-z| \leq \delta} \log^+ |\varphi_n(w)| d\sigma_w \leq \frac{1}{\pi \delta^2} \iint_{\Omega_\delta} \log^+ |\varphi_n(w)| d\sigma_w \leq \frac{\tilde{c}}{\pi \delta^2},$$

where in the first inequality we used the submean value property of subharmonic functions once more. So (φ_n) is uniformly bounded on compact subsets of K and therefore a normal family by Montel's theorem. \blacksquare

3.3 Proof of Theorem 1.6

Let F and G be maximal Blaschke products with critical sets $A := \mathcal{C}_1$ and $B := \mathcal{C}_2$ respectively. We fix a real number c with $0 < c < 1$. Then

$$\lambda_A(z) |dz| := \frac{c |F'(z)|}{1 - c^2 |F(z)|^2} |dz|, \quad \lambda_B(z) |dz| := \frac{c |G'(z)|}{1 - c^2 |G(z)|^2} |dz|$$

are two conformal pseudometrics on \mathbb{D} with zero set A resp. B and constant curvature -4 on $\mathbb{D} \setminus A$ and $\mathbb{D} \setminus B$ respectively. The product pseudometric

$$\lambda(z) |dz| := \lambda_A(z) \lambda_B(z) |dz|$$

has zero set $A \cup B$ (counting multiplicities) and a computation shows that

$$\kappa_\lambda(z) = -4 \left[\lambda_A(z)^{-2} + \lambda_B(z)^{-2} \right], \quad z \in \mathbb{D} \setminus (A \cup B). \quad (3.3)$$

We will show that there is a positive constant α such that

$$\kappa_\lambda(z) \leq -\alpha \quad \text{for all } z \in \mathbb{D} \setminus (A \cup B). \quad (3.4)$$

Fix $\xi \in \partial\mathbb{D}$. If $\xi \notin \overline{A}$, then, by Theorem 1.4, the maximal Blaschke product F has an analytic continuation to a neighborhood K_ξ of ξ . Thus $\lambda_A(z) |dz|$ is bounded on $K_\xi \cap \mathbb{D}$. In a similar way, we see that the density λ_B is bounded above near any point $\xi \in \partial\mathbb{D} \setminus \overline{B}$. Since by hypothesis $\overline{A} \cap \overline{B} \cap \partial\mathbb{D} = \emptyset$, we conclude from (3.3) that the curvature κ_λ is bounded above close to any point on $\partial\mathbb{D}$. Since $\kappa_\lambda(z) \rightarrow -\infty$ if $z \rightarrow z_0 \in A \cup B$, there is positive constant $\alpha > 0$ such that (3.4) holds.

As a result of the estimate (3.4), the conformal pseudometric

$$\mu(z) |dz| := \frac{\sqrt{\alpha}}{2} \lambda(z) |dz|$$

has curvature ≤ -4 on $\mathbb{D} \setminus (A \cup B)$ and zero set $A \cup B$. Corollary 2.7 shows that $A \cup B$ is an H^∞ critical set. ■

3.4 Proof of Theorem 1.7

Let F, B be two maximal Blaschke products with critical sets \mathcal{C}_F and \mathcal{C}_B , respectively. Consider the conformal pseudometric

$$\sigma(z) |dz| := ((B \circ F)^* \lambda_{\mathbb{D}})(z) |dz|,$$

so

$$\sigma(z) |dz| = \frac{|B'(F(z))| |F'(z)|}{1 - |B(F(z))|^2} |dz|.$$

In order to show that $B \circ F$ is a maximal Blaschke product, we appeal to Corollary 3.1, so we need to show that $\sigma \equiv \sigma_{\max}$, where $\sigma_{\max}(z) |dz|$ denotes the maximal conformal pseudometric on \mathbb{D} with zero set $\mathcal{C}_{B \circ F}$ and curvature -4 on $\mathbb{D} \setminus \mathcal{C}_{B \circ F}$. Recall that $\mathcal{C}_{B \circ F}$ denotes the critical set of the function $B \circ F$. We proceed in two steps.

1. *Step*: Assume B is a *finite* Blaschke product.

Clearly, $\sigma \leq \sigma_{max}$. In order to prove the reverse inequality we define the auxiliary function

$$s(z) := \log \frac{\sigma_{max}(z)}{\sigma(z)}, \quad z \in \mathbb{D}. \quad (3.5)$$

Since $\sigma_{max}(z) |dz|$ and $\sigma(z) |dz|$ do have the same zero set $\mathcal{C}_{B \circ F}$, we first note that $s : \mathbb{D} \rightarrow \mathbb{R}$ is continuous. Furthermore,

$$\Delta \left(\log \frac{\sigma_{max}(z)}{\sigma(z)} \right) = \Delta \log \sigma_{max}(z) - \Delta \log \sigma(z) = 4\sigma_{max}(z)^2 - 4\sigma(z)^2 \geq 0$$

for every $z \in \mathbb{D} \setminus \mathcal{C}_{B \circ F}$. Since $\mathcal{C}_{B \circ F}$ is a discrete subset of \mathbb{D} , we see that the continuous function $s : \mathbb{D} \rightarrow \mathbb{R}$ is subharmonic on \mathbb{D} and nonnegative by construction. It remains to prove that $s \equiv 0$.

For this purpose we consider the following conformal pseudometrics,

$$\lambda_{max}(z) |dz| := F^* \lambda_{\mathbb{D}}(z) |dz| = \frac{|F'(z)|}{1 - |F(z)|^2} |dz|$$

and

$$\mu_{max}(z) |dz| := B^* \lambda_{\mathbb{D}}(z) |dz| = \frac{|B'(z)|}{1 - |B(z)|^2} |dz|.$$

By Corollary 3.1, $\lambda_{max}(z) |dz|$ is the maximal conformal pseudometric on \mathbb{D} with zero set \mathcal{C}_F and curvature -4 on $\mathbb{D} \setminus \mathcal{C}_F$, and $\mu_{max}(z) |dz|$ is the maximal conformal pseudometric on \mathbb{D} with zero set \mathcal{C}_B and curvature -4 on $\mathbb{D} \setminus \mathcal{C}_B$.

In particular, if \tilde{B} denotes a finite Blaschke product with *zero set* \mathcal{C}_B , then the conformal pseudometric

$$|\tilde{B}(z)| \lambda_{\mathbb{D}}(z) |dz| = \frac{|\tilde{B}(z)|}{1 - |z|^2} |dz|$$

has curvature $-4|\tilde{B}(z)|^{-2} \leq -4$ on $\mathbb{D} \setminus \mathcal{C}_B$, so we get the crucial estimate

$$\mu_{max}(z) \geq \frac{|\tilde{B}(z)|}{1 - |z|^2} \quad \text{for any } z \in \mathbb{D}.$$

Since clearly, $\lambda_{max}(z) \geq \sigma_{max}(z) \geq \sigma(z)$ and since $\sigma(z) |dz|$ can be written in the form

$$\sigma(z) |dz| = \frac{\mu_{max}(F(z))}{\lambda_{\mathbb{D}}(F(z))} \cdot \lambda_{max}(z) |dz|$$

the above estimate for $\mu_{max}(z)$ gives

$$\lambda_{max}(z) \geq \sigma_{max}(z) \geq \sigma(z) \geq |\tilde{B}(F(z))| \lambda_{max}(z),$$

so rearranging terms yields

$$|\tilde{B}(F(z))| \leq \frac{\sigma(z)}{\lambda_{max}(z)} \leq \frac{\sigma_{max}(z)}{\lambda_{max}(z)} \leq 1. \quad (3.6)$$

We now make the obvious, but important observation that $\tilde{B} \circ F$ is a Blaschke product. This follows from the fact that F as a maximal Blaschke product is an *indestructible* Blaschke product and \tilde{B} is a finite Blaschke product. As a consequence of Frostman's characterization of Blaschke products (see [14, Ch. II, Theorem 2.4]), we get

$$\lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \log |\tilde{B}(F(re^{it}))| dt = 0,$$

which combined with (3.6) leads to

$$\frac{1}{2\pi} \int_0^{2\pi} \log \frac{\sigma_{max}(re^{it})}{\sigma(re^{it})} dt = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\sigma_{max}(re^{it})}{\lambda_{max}(re^{it})} dt - \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\lambda_{max}(re^{it})}{\sigma(re^{it})} dt \rightarrow 0$$

as $r \rightarrow 1$. Recalling the definition of the nonnegative subharmonic function $s : \mathbb{D} \rightarrow \mathbb{C}$ in (3.5), we finally deduce that $s \equiv 0$. This completes the proof that $B \circ F$ is a maximal Blaschke product for the case that B is a finite Blaschke product.

2. Step: Assume B is not a finite Blaschke product.

Let $\mathcal{C}_B = (z_j)$. Then for each positive integer n there exists a finite Blaschke product B_n with critical set $\mathcal{C}_{B_n} = (z_1, \dots, z_n)$, see Remark 2.6 (a). From the first step, we know that $B_n \circ F$ is a maximal Blaschke product with critical set $\mathcal{C}_{B_n \circ F} \subset \mathcal{C}_{B \circ F}$, so

$$\frac{|(B_n \circ F)'(z)|}{1 - |(B_n \circ F)(z)|^2} \geq \sigma_{max}(z) \geq \frac{|(B \circ F)'(z)|}{1 - |(B \circ F)(z)|^2}, \quad z \in \mathbb{D}.$$

Since Corollary 3.3 shows that the finite Blaschke products B_n converge locally uniformly in \mathbb{D} to B , we get

$$\sigma_{max}(z) = \frac{|(B \circ F)'(z)|}{1 - |(B \circ F)(z)|^2}, \quad z \in \mathbb{D},$$

as desired. ■

3.5 Proof of Theorem 1.8

Let B be a maximal Blaschke product with critical set \mathcal{C} and let $\zeta \in \partial\mathbb{D} \setminus \overline{\mathcal{C}}$. By Theorem 1.4, the function B has an analytic continuation across an open arc $\Gamma \subseteq \partial\mathbb{D}$ which contains ζ and such that $B(\Gamma) \subset \partial\mathbb{D}$. Hence, the so-called boundary Ahlfors' lemma (see Theorem 1.1 in [25]) shows that condition (1.2) holds for the boundary point ζ .

4 Extension to more general regions

In this section we extend Theorem 1.1 and Theorem 1.4 to more general subdomains of the complex plane than the unit disk. For this purpose we note that the extremal problem (1.1) of Theorem 1.1 has a natural counterpart for functions analytic and bounded in other domains than the unit disk:

Remark 4.1 (The Riemann mapping theorem and the Ahlfors mapping theorem)

Let $\Omega \subseteq \mathbb{C}$ be a domain containing the origin and let $\mathcal{C} = (z_j)$ be the critical set of a nonconstant function f in $H^\infty(\Omega)$. Here, $H^\infty(\Omega)$ is the set of all functions analytic and bounded in Ω . We denote by N the number of times that 0 appears in the sequence \mathcal{C} and let

$$\mathcal{F}_{\mathcal{C}}(\Omega) := \{f \in H^\infty(\Omega) : f'(z) = 0 \text{ for any } z \in \mathcal{C}\}.$$

Then, by a normal family argument, there is always at least one extremal function for the extremal problem

$$\sup \left\{ \operatorname{Re} f^{(N+1)}(0) : f \in \mathcal{F}_{\mathcal{C}}(\Omega), \|f\|_\infty \leq 1 \right\}. \quad (4.1)$$

For the special case $\mathcal{C} = \emptyset$, this extremal problem leads to the Riemann resp. Ahlfors mapping function:

(a) $\mathcal{C} = \emptyset$ and $\Omega \subsetneq \mathbb{C}$ simply connected

In this case the extremal problem (4.1) has a unique extremal function Ψ , namely the normalized Riemann map Ψ of the domain Ω , which maps Ω conformally onto \mathbb{D} such that $\Psi(0) = 0$ and $\Psi'(0) > 0$. In order to see this, we note that the standard textbook proof of existence of the Riemann map is through maximizing $\operatorname{Re} f'(0)$ over all *injective* functions in $\mathcal{F}_0(\Omega)$. That this apparently more restrictive extremal problem has the same solution as the original problem follows from the Schwarz lemma. To verify this, we first note that it is easily seen that the extremal function $\phi : \Omega \rightarrow \mathbb{D}$ to (4.1) is normalized by $\phi(0) = 0$ and $\phi'(0) > 0$. Since the normalized Riemann map Ψ of Ω belongs to $\mathcal{F}_0(\Omega)$, we get $\Psi'(0) \leq \phi'(0)$, so the holomorphic self-map $\omega := \Psi^{-1} \circ \phi$ of the unit disk fixes the origin and satisfies $\omega'(0) \geq 1$. The Schwarz lemma now implies that $\Psi^{-1} \circ \phi : \mathbb{D} \rightarrow \mathbb{D}$ is the identity function, so $\phi = \Psi$. We don't know of a *direct* way of proving that the extremal function ϕ for the extremal problem (4.1) is a conformal map from the domain Ω onto the unit disk \mathbb{D} .

(b) $\mathcal{C} = \emptyset$ and $\Omega \subsetneq \mathbb{C}$ a smooth multiply connected domain

If the domain Ω has finite connectivity $n \geq 2$, none of whose boundary components reduces to a point, then the extremal problem (4.1) has a unique extremal function, namely the Ahlfors map $\Psi : \Omega \rightarrow \mathbb{D}$. It is an $n : 1$ map from Ω onto \mathbb{D} such that $\Psi(0) = 0 < \Psi'(0)$; see [2], [16] and [5]. We also refer to [3] where the same extremal problem was considered in generality for various classes of analytic functions.

When Ω is a simply connected domain that is not equal to the whole complex plane, then Theorem 1.1 combined with the Riemann mapping theorem (Remark 4.1 (a)) leads to the following result about the extremal problem (4.1).

Theorem 4.2

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain containing the origin, let $\mathcal{C} = (z_j)$ be an $H^\infty(\Omega)$ critical set and let N denote the number of times that 0 appears

in the sequence \mathcal{C} . Then the extremal problem (4.1) has the unique extremal function $B_{\Psi(\mathcal{C})} \circ \Psi$. Here, Ψ is the normalized Riemann map for the domain Ω and $B_{\Psi(\mathcal{C})}$ is the extremal function for the extremal problem (1.1) for the critical set $\Psi(\mathcal{C})$ according to Theorem 1.1.

It would be interesting to study the extremal problem (4.1) in the presence of critical points when the domain Ω has connectivity $n > 1$, none of whose boundary components reduces to a point.

The next result is an immediate consequence of Theorem 1.1 and the Schwarz reflection principle.

Theorem 4.3

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain containing the origin and bounded by a real-analytic Jordan arc, and let \mathcal{C} be an $H^\infty(\Omega)$ critical set. Then the extremal function for the extremal problem (4.1) has an analytic continuation across any point of $\partial\Omega \setminus \overline{\mathcal{C}}$.

Perhaps the same result holds for any extremal function for the extremal problem (4.1) when Ω is a multiply connected domain with connectivity $n > 1$ and bounded by n real-analytic non-intersecting Jordan curves.

5 Further remarks and open problems

We close this paper with a number of remarks and further open problems.

Let us first return to Theorem 1.1. It says that every maximal Blaschke product is indestructible. This unavoidably suggests the following question.

Problem 5.1

Is every indestructible Blaschke product a maximal Blaschke product?

Note that an affirmative answer to Problem 5.1 would in particular imply that every locally univalent indestructible Blaschke product is in fact univalent.

The semigroup property of maximal Blaschke products (Theorem 1.7) states that if $B, C \in H^\infty$ are maximal Blaschke products then the composition $A := B \circ C$ is again a maximal Blaschke product. It is natural to ask for the converse statement: If $B, C \in H^\infty$ and $A = B \circ C$ is a maximal Blaschke product, does it follow that B and C are also maximal Blaschke products? The following simple observation provides a partial answer.

Proposition 5.2

Let $A, B, C \in H^\infty$ such that $A = B \circ C$ is a maximal Blaschke product. Then B is a maximal Blaschke product.

Proof. Let \tilde{B} be a maximal Blaschke product with critical set C_B and let $\lambda_{max}(z) |dz|$ denote the maximal conformal pseudometric on \mathbb{D} with zero set C_B and curvature -4 on $\mathbb{D} \setminus C_B$. Then, by Corollary 3.1,

$$\lambda_{max}(w) = \frac{|\tilde{B}'(w)|}{1 - |\tilde{B}(w)|^2} \geq \frac{|B'(w)|}{1 - |B(w)|^2} \quad \text{for every } w \in \mathbb{D}.$$

This inequality holds in particular for any $w = C(z)$, $z \in \mathbb{D}$, so by multiplying both sides with $|C'(z)|$, we get for every $z \in \mathbb{D}$

$$C^* \lambda_{max}(z) = \frac{|(\tilde{B} \circ C)'(z)|}{1 - |(\tilde{B} \circ C)(z)|^2} \geq \frac{|(B \circ C)'(z)|}{1 - |(B \circ C)(z)|^2} = \frac{|A'(z)|}{1 - |A(z)|^2} \geq C^* \lambda_{max}(z),$$

where the last inequality comes from the maximality of A . Hence equality holds throughout and Liouville's Theorem implies $B \circ C = T \circ \tilde{B} \circ C$ for some unit disk automorphism T , i.e., $B = T \circ \tilde{B}$. Thus B is a maximal Blaschke product. \square

Problem 5.3

Let $A, B, C \in H^\infty$ such that $A = B \circ C$ is a maximal Blaschke product. Does it follow that C is a maximal Blaschke product?

Remark 5.4

It is well-known and easy to prove that if the function $A = B \circ C$ in Problem 5.3 is a *finite* Blaschke product, then both B and C are finite Blaschke products. Hence, if the answer to Problem 5.3 is affirmative, then it would be interesting to explore the possibility of a “prime factorization” of maximal Blaschke products in a way similar to the recent extension of Ritt's celebrated factorization theorem for finite Blaschke products due to Ng and Wang (see [32]). In this connection, information about the critical *values* of maximal Blaschke products would be valuable.

Remark 5.5

In view of Problem 5.1 and the discussion above, the following two questions arise.

- (a) Is the composition of two indestructible Blaschke products B and C an indestructible Blaschke product?
- (b) If $B, C \in H^\infty$ and $B \circ C$ is an indestructible Blaschke product, must also B and C be indestructible Blaschke products?

These issues will be discussed in the forth-coming paper [26].

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